JOURNAL OF APPROXIMATION THEORY 21, 375–384 (1977)

On Optimal Polynomial Interpolation of Analytic Functions*

DAVID L. BARROW

Department of Mathematics, Texas A & M University, College Station, Texas 77801

Communicated by E. W. Cheney

Received March 31, 1976

This paper discusses the problem of choosing the Lagrange interpolation points $T = (t_0, t_1, ..., t_n)$ in the interval $-1 \leqslant t \leqslant 1$ to minimize the norm of the error, considered as an operator from the Hardy space $H^2(R)$ of analytic functions to the space $C[-1, 1]$. It is shown that such optimal choices converge for fixed n, as $R \to \infty$, to the zeros of a Chebyshev polynomial. Asymptotic estimates are given for the norm of the error for these optimal interpolations, as $n \to \infty$ for fixed R. These results are then related to the problem of choosing optimal interpolation points with respect to the Eberlein integral. This integral is based on a probability measure over certain classes of analytic functions, and is used to provide an average interpolation error over these classes. The Chebyshev points are seen to be limits of optimal choices in this case also.

1. INTRODUCTION

We consider two ways of estimating the error involved in approximating certain classes of analytic functions by interpolating polynomials. The first is analogous to a method introduced by Davis [3] for linear functionals, and is essentially the norm of the error considered as an operator from one Banach space to another. Specifically, let P_T denote the operator of polynomial interpolation at the points $T = (t_0, t_1, ..., t_n)$ in the interval $[-1, 1]$, and let $E_T = I - P_T$. If $H^2(R)$ denotes the Hardy space of functions analytic in the disk $|z| < R$, $R > 1$, and $C[-1, 1]$ is the set of continuous functions on $[-1, 1]$ with the supremum norm, then $\| E_T \|_R$ will denote the operator norm of $E_T: H^2(R) \to C[-1, 1]$. We discuss the problem of choosing T to minimize $\| E_T \|_R$, and denote such a choice by T_R. Theorem 1 states that T_R tends to the zeros of the Chebyshev polynomial of degree $n + 1$ as $R \to \infty$. Theorem 2 gives asymptotic estimates for $\| E_{T_R} \|_R$, for fixed R, as $n \to \infty$.

The second way of measuring the error uses the Eberlein integral [4], and provides an average or root-mean-square error over a class of functions

* This work was partially supported by the NSF under Grant DCR75-04545.

ISSN 0021-9045

analytic in the disk $|z| < R$, $R > 1$. If we denote this error by $\|E_T\|_{\sigma_R}$, it turns out that $\|E_T\|_{\sigma_R} = (2\pi R/3^{1/2})^{1/2} \|E_T\|_{R3^{1/2}}$. It follows immediately that if $\tilde{T}_R$ denotes a choice of interpolation points which minimizes $\|E_T\|_{\sigma_R}$, then $\tilde{T}_R = T_{R3^{1/2}}$. Hence, Theorems 1 and 2 can be applied to this case also.

In Section 2 we discuss the error associated with the classes $H^2(R)$, and state Theorems 1 and 2. Section 3 is devoted to the error associated with the Eberlein integral. Numberical results related to optimum interpolation (relative to the Eberlein integral) are discussed in Section 4.

2. Optimal Interpolation Points for $H^2(R)$

In [3] Davis introduced the idea of expressing bounds for the error in certain numerical approximations as

$$|E(f)| \le \|E\| \, \|f\|$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is in the Hardy space H^2 of functions analytic in $|z| < 1$ with

$$\|f\|^2 = 2\pi \sum_{k=0}^{\infty} |a_k|^2 < \infty,$$

and E is an error linear functional. Haber [6] and Valentin [10] have used this approach for the class $H^2(R)$ of functions analytic in $|z| < R$, $R > 1$, such that

$$\|f\|_R^2 = \int_{|z|=R} |f(z)|^2 \, ds = 2\pi R \sum_{k=0}^{\infty} |a_k|^2 R^{2k} < \infty.$$

$H^2(R)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{|z|=R} f(z) \, \overline{g(z)} \, ds$$

and orthonormal basis

$$\phi_k(z) = (2\pi R)^{-1/2} (z/R)^k, \qquad k = 0, 1, 2, \ldots .$$

$H^2(R)$ has the Szego reproducing kernel

$$K(z, w) = \sum_{k=0}^{\infty} \phi_k(z) \, \overline{\phi_k(w)} = \frac{R}{2\pi} \frac{1}{R^2 - z\bar{w}}$$

such that

$$f(w) = \int_{|z|=R} K(z, w) \, f(z) \, ds$$

for $f \in H^2(R)$, $|w| < R$. If E is in the dual of $H^2(R)$, we denote its norm by $\| E \|_R$; if E is *real* (that is, $\overline{E(f)} = E(\bar{f})$), then

$$\| E \|_R^2 = \sum_{k=0}^{\infty} | E(\phi_k)|^2 = \sum_{k=0}^{\infty} E^{(z)}\phi_k(z) \cdot \overline{E^{(w)}\phi_k(w)}$$
$$= E^{(z)}E^{(w)}K(z, w), \tag{1}$$

where the superscript z or w indicates with respect to which variable E is acting.

Let $C[-1, 1]$ be the Banach space of sup-normed continuous functions on $-1 \leqslant t \leqslant 1$. If $A: H^2(R) \to C[-1, 1]$ is a bounded linear operator, we denote its norm by $\| A \|_R$ (the specific usage of $\| \ \|_R$ should make it clear whether it is the norm of an element in $H^2(R)$, a linear functional, or an operator).

Let n be a fixed nonnegative integer. We denote by Δ^n the subset of Euclidean space R^{n+1}

$$\Delta^n = \{T = (t_0, t_1, \ldots, t_n): -1 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1\},$$

with Euclidean norm denoted by $| T |$. For $T \in \Delta^n$, let P_T: $H^2(R) \to C[-1, 1]$ be the operator of polynomial interpolation at the points of T; that is (in the notation of [2, p. 225]),

$$P_T f(t) = \sum_{i=0}^{n} f[t_0, t_1, \ldots, t_i] \prod_{j=0}^{i-1} (t - t_j).$$

Let E_T be the error operator $I - P_T$, and $E_{T,t}$ the error linear functional at t. Thus (cf. [2, pp. 225, 231])

$$E_{T,t}(f) = E_T f(t) = f[t_0, t_1, \ldots, t_n, t] \prod_{i=0}^{n} (t - t_i) = \frac{f^{(n+1)}(\xi_t)}{(n+1)!} \prod_{i=0}^{n} (t - t_i). \tag{2}$$

If the points of T are distinct, it will be convenient to use the expression

$$E_{T,t}(f) = f(t) - \sum_{i=0}^{n} f(t_i)\, l_i(t) = \sum_{i=0}^{n} (f(t) - f(t_i))\, l_i(t), \tag{3}$$

where

$$l_i(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{n} \frac{(t - t_j)}{(t_i - t_j)}.$$

$E_{T,t}$ is a real linear functional, and $E_{T,t}(f)$ depends continuously on $(T, t) \in \Delta^n \times [-1, 1]$. Hence, by (1), $\| E_{T,t} \|_R$ also depends continuously on (T, t).

We let C_n denote the "Chebyshev points"; that is, $C_n = (c_0, ..., c_n) \in \Delta^n$ will be the zeros of the Chebyshev polynomial $P_{n+1}(t)$, where $P_{n+1}(\cos\theta) = \cos((n+1)\theta)$. Thus,

$$c_k = -\cos\left(\frac{2k+1}{2(n+1)}\pi\right), \qquad k = 0, 1, ..., n.$$

Now consider the problem of choosing T to minimize $\| E_T \|_R$. Since $\| E_T \|_R$ is a continuous function of T, there is a choice of T in the compact set Δ^n, which we denote by T_R, such that

$$\| E_{T_R} \|_R \leq \| E_T \|_R \qquad \text{for all} \quad T \in \Delta^n.$$

As was done in [10] (and similarly in [1, 11]) for quadrature formulas, we ask how T_R behaves for large R. The answer is that letting R increase tends to emphasize the importance of the error on ϕ_k as compared with ϕ_{k+1}. In the limit as $R \to \infty$, the optional choice of T minimizes the error on ϕ_{n+1} (it is already zero for $\phi_0, \phi_1, ..., \phi_n$). Precisely, we have

THEOREM 1. *If $\{T_R\}_{R>1}$ is any set of optimal T's (as defined above), then $T_R \to C_n$ as $R \to \infty$.*

Proof. For any $R > 1$ and $T \in \Delta^n$,

$$\| E_T \|_R = \sup_{\|f\|_R=1} \sup_{-1\leqslant t\leqslant 1} | E_T f(t) | = \sup_{-1\leqslant t\leqslant 1} \sup_{\|f\|_R=1} | E_T f(t) | = \sup_{-1\leqslant t\leqslant 1} \| E_{T,t} \|_R. \quad (4)$$

Now since $E_{T,t} \in H^2(R)^*$, its norm is given by

$$\| E_{T,t} \|_R^2 = \sum_{k=0}^{\infty} | E_{T,t}(\phi_k) |^2 = \sum_{k=n+1}^{\infty} \frac{| E_{T,t}(z^k) |^2}{2\pi R^{2k+1}}.$$

Let

$$\phi_{T,R}(t) = 2\pi R^{2n+3} \| E_{T,t} \|_R^2 = | E_{T,t}(z^{n+1}) |^2 + \sum_{k=1}^{\infty} \frac{| E_{T,t}(z^{n+k+1}) |^2}{R^{2k}}.$$

It is clear by (4) that choosing $T \in \Delta^n$ to minimize

$$\| \phi_{T,R} \| = \sup_{-1\leqslant t\leqslant 1} | \phi_{T,R}(t) |$$

will produce T_R, a minimizer of $\| E_T \|_R$. Using (2), it follows that $\phi_{T,R}$ can be expressed as

$$\phi_{T,R}(t) = \psi_T(t)[1 + F(t, T, R)],$$

where

$$\psi_T(t) = \prod_{i=0}^{n} (t - t_i)^2$$

and,

$$| F(t, T, R)| \leqslant \sum_{k=1} \binom{n+k+1}{n+1}^2 \Big/ R^{2k} \equiv F(R).$$

It is routine to verify that the above series converges for each $R > 1$ (for example, by comparing it to the convergent integral $\int_0^\infty (x + n + 1)^{2(n+1)} \times R^{-2x}\, dx$). It follows that $F(R) \to 0$ as $R \to \infty$, and so we have shown that $\phi_{T,R}(t) \to \psi_T(t)$, uniformly for $(T, t) \in \Delta^n \times [-1, 1]$, as $R \to \infty$.

We suppose now that the theorem is false. Then there is a $\delta > 0$ so that for arbitrarily large R, $| T_R - C_n | \geqslant \delta$. It is well known that $\| \psi_{C_n} \| < \| \psi_T \|$ for all $T \in \Delta^n$, $T \neq C_n$, where the norm is that of $C[-1, 1]$ (cf. [7, p. 31, Theorem 11]). If we let K be the compact set $\Delta^n \backslash \{T\colon | T - C_n | < \delta\}$, then there is a $\tilde{T} \in K$ such that $\| \psi_{\tilde{T}} \| \leqslant \| \psi_T \|$ for all $T \in K$. Let $\epsilon = \| \psi_{\tilde{T}} \| - \| \psi_{C_n} \| > 0$. Let R be such that $\| \phi_{T,R} - \psi_T \| < \epsilon/3$ for all $T \in \Delta^n$, while also $T_R \in K$. Then

$$\| \phi_{C_n,R} \| \leqslant \| \phi_{C_n,R} - \psi_{C_n} \| + \| \psi_{C_n} \| < \epsilon/3 + \| \psi_{C_n} \|$$

and

$$\| \phi_{T_R,R} \| > \| \psi_{T_R} \| - \| \phi_{T_R,R} - \psi_{T_R} \| > \| \psi_{\tilde{T}} \| - \epsilon/3.$$

It follows that $\| \phi_{T_R,R} \| > \| \phi_{C_n,R} \| + \epsilon/3$, which contradicts the choice of T_R. This completes the proof of Theorem 1.

We now shift the emphasis and consider, for fixed $R > 1$, the asymptotic behavior of minimum norm interpolation formulas as $n \to \infty$. The following theorem is similar to one proved in [6] for quadrature formulas. Let the optimal T_R described above now be denoted by $T_{R,n}$, to show explicitly the dependence on n.

THEOREM 2. *Let $R > 1$ and $\bar{\rho} = R + (R^2 - 1)^{1/2}$. Then for any ρ, $1 < \rho < \bar{\rho}$, there is a positive constant $A(\rho)$ such that*

$$\| E_{T_{R,n}} \|_R \leqslant \| E_{C_n} \|_R \leqslant A(\rho)\, \rho^{-n}. \tag{5}$$

Proof. As in [7], we let D_ρ be the interior of the ellipse E_ρ having foci at ± 1 and the sum of the half-axes ρ; that is, E_ρ is the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $a = (\rho + \rho^{-1})/2$ and $b = (\rho - \rho^{-1})/2$. Let f be in $H^2(R)$. Then f is analytic in $D_{\bar{\rho}}$, which is the largest such ellipse contained in the open disk $| z | < R$. The formula $f(w) = \int_{|z|=R} K(z, w) \overline{f(z)}\, ds$ and the Cauchy–Schwarz inequality yield

$$\max_{z \in E_\rho} | f(z)| \leqslant M_\rho \| f \|_R\,. \tag{6}$$

Following [7], we let $E_n(f)$ denote the distance (in $C[-1, 1]$) from f to the space P_n of polynomials of degree not exceeding n; that is,

$$E_n(f) = \inf_{p_n \in P_n} \sup_{-1 \leqslant t \leqslant 1} |f(t) - p_n(t)|.$$

As in the proof of Theorem 7, p. 76 of [7], there is a constant $M(\rho)$ so that

$$E_n(f) \leqslant M(\rho)\, \rho^{-n}(\max_{z \in E_\rho} |f(z)|),$$

and so

$$E_n(f) \leqslant N(\rho)\, \rho^{-n} \|f\|_R . \tag{7}$$

If we let $\|E_{C_n}\| = \|I - P_{C_n}\|$ denote the norm of E_{C_n} as an operator from $C[-1, 1]$ into itself, then it is known (cf. [8, p. 263]) that

$$\max_{-1 \leqslant t \leqslant 1} |E_{C_n} f(t)| \leqslant \|E_{C_n}\| E_n(f) \tag{8}$$

and that [8, p. 283]

$$\|E_{C_n}\| \leqslant K_1 + K_2 \log n. \tag{9}$$

To verify (5), it is enough to show $\|E_{C_n}\|_R\, \rho^n \to 0$ as $n \to \infty$. For this, take $\rho < \rho_1 < 1$ and combine Eqs. (7) to (9), with ρ replaced by ρ_1, to obtain

$$\|E_{C_n}\|_R \leqslant (K_1 + K_2 \log n)\, N(\rho_1)\, \rho_1^{-n} = (K_1 + K_2 \log n)\, N(\rho_1)(\rho/\rho_1)^n \rho^{-n}$$

and the desired convergence follows.

3. A Connection with the Eberlein Integral

In [4] Eberlein introduced an integral over the unit sphere S_∞ of the (real) sequence space l_1. Thus, $x = \{x_k\}_{k=0}^\infty \in S_\infty$ if and only if $\sum_{k=0}^\infty |x_k| \leqslant 1$. The integral is induced by a positive linear functional on $C(S_\infty)$, the weak-$*$ continuous real-valued functions on S_∞, and is denoted

$$E(f) = \int_{S_\infty} f(x)\, d_E x.$$

Eberlein shows that the integral can be extended to elements in the dual of l_1, i.e., bounded sequences $y = \{y_k\}$, and that

$$\int_{S_\infty} \langle x, y \rangle\, d_E x = 0$$

and

$$\int_{S_\infty} \langle x, y\rangle^2 \, d_E x = \sum_{k=0}^{\infty} \frac{y_k^2}{3^{k+1}} = \sum_{k=0}^{\infty} \frac{\langle e_k, y\rangle^2}{3^{k+1}},$$

where e_k, $k = 0, 1,...$ is the natural basis for l_1. In fact, the latter equation holds whenever the series converges.

We now apply this integral to obtain a measure of the "average" error (more precisely, "root-mean-square" or rms error) of a numerical approximation over certain function spaces. For $R > 1$, let A_R be the set of real functions

$$x(t) = \sum_{k=0}^{\infty} x_k (t/R)^k, \qquad -1 \leqslant t \leqslant 1,$$

with $\| x \| = \sum_{k=0}^{\infty} | x_k | < \infty$. For λ a linear functional on A_R, we define the *variance of* λ as

$$\sigma_R^2(\lambda) = \int_{S^\infty} \langle x, \lambda\rangle^2 \, d_E x = \sum_{k=0}^{\infty} (\lambda_k^2/3^{k+1})$$

where $\lambda_k = \lambda(t^k/R^k)$.

Sarma and Stroud [9] have used $\sigma_R(\lambda)$ to give a measure of the average error over the unit ball of A_R when $R = 1$ and λ is the error functional of a one-dimensional quadrature formula. Their results include generalizations to functions of more than one variable.

We note the following equality:

$$\sigma_R^2(\lambda) = \sum_{k=0}^{\infty} \frac{| \lambda(t^k)|^2}{R^{2k}\, 3^{k+1}} = \frac{2\pi R}{3^{1/2}} \sum_{k=0}^{\infty} \frac{| \lambda(t^k)|^2}{2\pi(3^{1/2}R)^{2k+1}} = \frac{2\pi R}{3^{1/2}} \| \lambda \|_{3^{1/2}R}^2, \quad (10)$$

when λ is considered as an element of the dual of $H^2(3^{1/2}R)$. Haber [5] pointed out this connection in the case $R = 1$. Now let $\lambda = E_{T,t}$, the error functional considered earlier. In view of (10), if $\tilde{T}_R \in \Delta^n$ minimizes $\sup_{-1 \leqslant t \leqslant 1} \sigma_R(E_{T,t})$, then $\tilde{T}_R = T_{3^{1/2}R}$, where $T_{3^{1/2}R}$ minimizes the norm $\| E_T \|_{R3^{1/2}}$ considered in the last section. It thus follows as a corollary to Theorem 1 that $\tilde{T}_R \to C_n$ as $R \to \infty$. We may intrepret this result as follows. The Chebyshev points C_n are an optimal choice (relative to the Eberlein measure) for interpolating real power series having an infinite radius of convergence, in the sense that they are the limit, as $R \to \infty$, of choices $\tilde{T}_R$ which minimize the maximum (over $[-1, 1]$) rms error over the unit sphere of the space A_R.

4. Numerical Results

In this section we give computational results related to optimal interpolations (relative to the Eberlein measure) for the space A_1. This is the space to which Eberlein originally applied his integral, his motivation being applications to numerical integration. A_1 is the space of real power series

$$x(t) = \sum_{k=0}^{\infty} x_k t^k, \qquad -1 \leqslant t \leqslant 1, \qquad \| x \| = \sum_{k=0}^{\infty} | x_k | < \infty.$$

If λ is a linear functional on this space, we let

$$\sigma^2(\lambda) = \sum_{k=0}^{\infty} \frac{|\lambda(t^k)|^2}{3^{k+1}}$$

and call $\sigma(\lambda)$ the root-mean-square of λ (σ is the σ_1 of the last section). For given n, let $E_{T,t}$ be the linear functional considered before. We consider the problem of choosing T to minimize

$$\max_{-1 \leqslant t \leqslant 1} \sigma(E_{T,t}) = \| E_T \|_\sigma .$$

Denote such a choice by T_n^*. As the computation involved in approximating T_n^* numerically is considerable, we limit ourselves to finding optimal symmetric formulas for the cases $n = 1$ and 2.

Case (i). $n = 1$, $T = (-a, a)$, $a > 0$. A computation based on (3) gives

$$\sigma^2(E_{T,t}) = \frac{(9 + a^2t^2)(t^2 - a^2)^2}{(9 - a^4)(9 - a^2t^2)(3 - t^2)} .$$

This function is zero only at $t = \pm a$, and has the least maximum value on $[-1, 1]$ when its values at 0 and ± 1 are equal. This leads to the equation

$$5a^6 + 3a^4 - 51a^2 + 27 = 0.$$

Solving numerically, we obtain $a \simeq 0.75235573$, and for this choice of T, $\| E_T \|_\sigma \simeq 0.11092667$.

Case (ii). $n = 2$, $T = (-a, 0, a)$, $a > 0$. In this case,

$$\sigma^2(E_{T,t}) = \frac{(t^2 - a^2)^2\, t^2 (9 + a^2t^2)}{3(3 - t^2)(9 - a^2t^2)(9 - a^4)} .$$

The secant method was applied to the derivative of this function to find the local maximum between 0 and a, and an iteration on a was used to make this local maximum equal to $\sigma^2(E_{T,1})$. The results are $a \simeq 0.88591934$, and for this choice of T, $\| E_T \|_\sigma \simeq 0.033105735$.

We next obtain computational estimates for $\| E_{T_n^*} \|_\sigma$ by obtaining them for $\| E_{C_n} \|_\sigma$. Such estimates might be useful, as pointed out in [9], by applying the Chebyshev inequality of probability theory: If $x(t)$ is chosen at random in the unit ball of A_1, then for any $t \in [-1, 1]$ and $\rho > 0$, the probability that

$$| E_{T,t}(x)| \leqslant \rho\sigma(E_{T,t}),$$

and therefore that $\sup_{-1 \leqslant t \leqslant 1} | E_{T,t}(x)| \leqslant \rho \| E_T \|_\sigma$, is greater than $1 - \rho^{-2}$. Also, if $x \in H^2(3^{1/2})$ and we happen to know $\| x \|_{3^{1/2}}$, then

$$\| E_T x \| \leqslant \| E_T \|_{3^{1/2}} \| x \|_{3^{1/2}} = \left(\frac{3^{1/2}}{2\pi}\right)^{1/2} \| E_T \|_\sigma \| x \|_{3^{1/2}},$$

the last equality following from (10) with $R = 1$ and $\lambda = E_{T,t}$.

We have computed approximations to $\| E_{C_n} \|_\sigma$ for $n = 2, 3,..., 14$. The results are believed to be accurate to the number of digits given. The computation proceeded as follows: Let

$$\begin{aligned} q_n(t) = \sigma^2(E_{C_{n,t}}) &= \sum_{k=0}^{\infty} \frac{E_{C_{n,t}}(t^k)^2}{3^{k+1}} \\ &= \sum_{i,j=0}^{n} l_i(t)\, l_j(t) \left[\frac{1}{3 - t^2} - \frac{1}{3 - tt_i} - \frac{1}{3 - tt_j} + \frac{1}{3 - t_i t_j}\right]. \end{aligned}$$

By symmetry it was sufficient to consider the interval $0 \leqslant t \leqslant 1$. The local maxima of $q_n(t)$ between adjacent t_i's were found by applying the secant method to find zeros of $q_n'(t)$. Then the local maxima were searched to find the maximum of $q_n(t)$. In every case, this occurred at $t = 1$. Furthermore, the local maxima increased monotonically away from the origin.

In Table I the column of ratios appears to be consistent with the following

COROLLARY TO THEOREM 2. *If* $1 < \rho < 3^{1/2} + 2^{1/2} \simeq 3.14626437$, *then there is a constant* $B(\rho)$ *such that*

$$\| E_{C_n} \|_\sigma \leqslant B(\rho)\, \rho^{-n}.$$

Proof. By (10), with $\lambda = E_{C_{n,t}}$ and $R = 1$, we obtain

$$\| E_{C_n} \|_\sigma = (2\pi/3^{1/2})^{1/2} \| E_{C_n} \|_{3^{1/2}}.$$

Now apply Theorem 2 with $R = 3^{1/2}$.

All computations were carried out on the AMDAHL 470 computer at the Data Processing Center at Texas A & M University, using double-precision arithmetic in FORTRAN which carries about 16 significant digits.

TABLE I

Values of $\| E_{C_n} \|_\sigma$ for the Chebyshev Points C_n

n	$\| E_{C_n} \|_\sigma$	$\| E_{C_{n-1}} \|_\sigma / \| E_{C_n} \|_\sigma$
2	0.1263 5817	
3	0.3819 7314 (−1)	3.308038
4	0.1159 3467 (−1)	3.294728
5	0.3535 5323 (−2)	3.279129
6	0.1082 7846 (−2)	3.265222
7	0.3328 5138 (−3)	3.253057
8	0.1026 5325 (−3)	3.242483
9	0.3174 8522 (−4)	3.233324
10	0.9843 2640 (−5)	3.225406
11	0.3058 2783 (−5)	3.218564
12	0.9519 9379 (−6)	3.212498
13	0.2968 3129 (−6)	3.207188
14	0.9271 9081 (−7)	3.201406

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